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Representations of coherent and squeezed states in a f-deformed Fock space

R Roknizadeh and M K Tavassoly

Department of Physics, University of Isfahan, Isfahan, Iran

E-mail: rokni@sci.ui.ac.ir and mk.tavassoly@sci.ui.ac.ir

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Abstract
We establish some of the properties of the states interpolating between number and coherent states denoted by $|n\rangle_\lambda$; among them are the reproduction of these states by the action of an operator-valued function on $|n\rangle$ (the standard Fock space) and the fact that they can be regarded as f-deformed coherent bound states. In this paper we use them as the basis of our new Fock space which in this case is not orthogonal but normalized. Then by some special superposition of them we obtain new representations for coherent and squeezed states in the new basis. Finally the statistical properties of these states are studied in detail.

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1. Introduction

As is well known the Hilbert space is a natural framework for the mathematical description of many areas of physics: certainly for quantum mechanics (and quantum field theory), signal and image analysis [Lynn(1986)], etc. In all these cases any arbitrary vector can be represented in terms of the elements of some basis $\{|n\rangle, n \in \mathbb{N}\}_{n=0}^{\infty}$. An orthonormal basis is the one that is mostly advocated by mathematicians. But unfortunately, orthonormal bases are sometimes difficult to find and hard to work with. So the generalization to non-orthogonal basis vectors (such as that we will introduce in this paper denoted by $\{|n\rangle_\lambda, n \in \mathbb{N}\}_{n=0}^{\infty}$) has been proposed. These vectors have properties such as fast convergence, uniqueness of decomposition, etc. The resulting object is called a '*frame*' [Ali(1993)]; or more precisely a '*discrete frame*'. The non-orthogonal states are also used as a tool in '*generalized measurement*' that is a very important subject in view of the foundations of quantum mechanics, quantum nondemolition measurement and quantum information theory [Peres(1995)]. This paper, which contains the construction of coherent states (CSs) and squeezed state (SSs) using a special set of non-orthogonal but normalizable bases $\{|n\rangle_\lambda, n \in \mathbb{N}\}_{n=0}^{\infty}$ instead of orthonormal ones $\{|n\rangle, n \in \mathbb{N}\}_{n=0}^{\infty}$, can be classified in the discrete frame background. Thus in our opinion this approach provides a more fundamental and certainly more flexible rule than that of the usual one considered extensively in the physical literature for construction of new generalized CSs, a concept which has attained the most applications in quantum optics, as well as other fields of physics [Klauder(1985), Perelomov(1986)].

For a simple harmonic oscillator (SHO) in natural units and with unit frequency we have

$$H = a^\dagger a + \tfrac{1}{2} \qquad [a, a^\dagger] = 1 \qquad (a^\dagger)^\dagger = a \tag{1}$$

where a, $a^\dagger$ and H are standard annihilation, creation and Hamiltonian operators, respectively. Following the particular kind of deformation proposed by Beckers *et al* [Beckers(1998)] we consider the parametric harmonic oscillator by deforming the creation operator in such a way that

$$a_\lambda^\dagger = a^\dagger + \lambda I \qquad \lambda \in \mathbb{R} \tag{2}$$

without changing the annihilation operator. So the λ-Hamiltonian becomes

$$H_\lambda = a_\lambda^\dagger a + \tfrac{1}{2}. \tag{3}$$

Note that $\left(a_\lambda^\dagger\right)^\dagger \neq a_\lambda$, $H_\lambda^\dagger \neq H_\lambda$, but as for SHO yet we have

$$\left[a, a_\lambda^\dagger\right] = 1 \qquad \left[H_\lambda, a_\lambda^\dagger\right] = a_\lambda^\dagger. \tag{4}$$

To manifest our motivation for considering the special kind of deformation in equation (2) we offer the following discussion. In a previous paper we have shown that a large class of generalized CSs can be obtained by changing the basis in the underlying Hilbert space. We have presented a systematic formalism that the particular deformation which we employed in this paper is a special case of the general scheme which has been introduced in [Ali(2004)]. To clarify further, we will explain briefly the setting. Let $\mathcal{H}$ be a Hilbert space and T, T^{-1} be operators densely defined and closed on $\mathcal{D}(T)$ and $\mathcal{D}(T^{-1})$, respectively, and $F = T^\dagger T$. Two new Hilbert spaces $\mathcal{H}_F$, $\mathcal{H}_{F^{-1}}$ are the completions of the sets $\mathcal{D}(T)$ and $\mathcal{D}(T^{\dagger^{-1}})$ with the scalar products

$$\langle f|g\rangle_F = \langle f|Fg\rangle_{\mathcal{H}} \qquad \langle f|g\rangle_{F^{-1}} = \langle f|F^{-1}g\rangle_{\mathcal{H}} \tag{5}$$

respectively. Considering the generators of the Weyl–Heisenberg algebra as bases on $\mathcal{H}$, we may obtain the transformed generators on $\mathcal{H}_F$ such as

$$a_F = T^{-1} a T \qquad a_F^\dagger = T^{-1} a^\dagger T \qquad N_F = T^{-1} N T. \tag{6}$$

A similar argument may be followed for the Hilbert space $\mathcal{H}_{F^{-1}}$, whose properties were discussed in [Ali(2004)]. It is obvious that the oscillator algebra remains unchanged. Now choosing $T = \exp(-\lambda a)$ and using (6) for the annihilation and creation operators on $\mathcal{H}_F$ we recover the exact forms of the deformation introduced by Fu *et al* [Fu(2000)] and Beckers *et al* [Beckers(1998)] which are employed in this paper, i.e.

$$a_F = a \qquad a_F^\dagger = a^\dagger + \lambda I. \tag{7}$$

Other selections for the T-operator lead to other families of generalized CSs. In this manner we have established the basic foundation of the particular kind of deformation we have used in this work, in the general theory of constructing the CSs in an non-orthogonal basis. The case in which the annihilation operator is deformed as $a_\lambda = a + \lambda I$, has also been already considered and discussed by Ali and us in [Ali(2004)].

The eigenvalue equation

$$H_\lambda |n\rangle_\lambda = E_{n,\lambda} |n\rangle_\lambda \tag{8}$$

has been solved in [Beckers(1998)], and led to the λ-states

$$\psi_{n,\lambda}(x) = 2^n n! \pi^{-1/4}\, \mathrm{e}^{-x^2/2} L_n^{(0)}(-\lambda^2) H_n(x + \lambda/\sqrt{2}) \tag{9}$$

where $L_n^{(0)}$ and H_n are, respectively, the well-known Laguerre and Hermite polynomials of order n and H_λ is isospectral with H_{SHO}, i.e.

$$E_{n,\lambda} = E_n = n + \tfrac{1}{2} \qquad n = 0, 1, 2, \ldots. \tag{10}$$

Also the operation of a and $a_\lambda^\dagger$ on λ-states is as follows:

$$a|n\rangle_\lambda = \sqrt{n}\left(\frac{L_{n-1}^{(0)}(-\lambda^2)}{L_n^{(0)}(-\lambda^2)}\right)^{1/2}|n-1\rangle_\lambda \tag{11}$$

$$a_\lambda^\dagger|n\rangle_\lambda = \sqrt{n+1}\left(\frac{L_{n+1}^{(0)}(-\lambda^2)}{L_n^{(0)}(-\lambda^2)}\right)^{1/2}|n+1\rangle_\lambda. \tag{12}$$

The states introduced in equation (9) are not orthogonal and their inner products read

$${}_\lambda\langle m|n\rangle_\lambda = \left[L_m^{(0)}(-\lambda^2)L_n^{(0)}(-\lambda^2)\right]^{-1/2}\sum_k \frac{\lambda^{2k+m-n}(n!m!)^{1/2}}{k!(n-k)!(m-n+k)!} \tag{13}$$

but it is trivial from equation (13) that they are normalized, i.e. ${}_\lambda\langle n|n\rangle_\lambda = 1$.

2. Some remarkable points

Before we start our main discussion, let us list some of the interesting and important points we may conclude. The following results may be considered as complementary in regard to the previous related works mentioned earlier [Fu(2000), Beckers(1998)].

(a) It can easily be shown that the states $|n\rangle_\lambda$ can be regarded as the basis for our new Fock space, since the necessary and sufficient conditions mentioned for a one-dimensional quantum mechanical Fock space in [Bardek(2000)] will be satisfied. In our problem these conditions are briefly (i) the existence of a vacuum state such that $a|0\rangle = 0$, (ii) $\langle 0|aa_\lambda^\dagger|0\rangle > 0$, (iii) $\left[aa_\lambda^\dagger, a_\lambda^\dagger a\right] \neq 0$ and $aa_\lambda^\dagger \neq a_\lambda^\dagger a$.

(b) Despite the appearance of equation (13), ${}_\lambda\langle m|n\rangle_\lambda = {}_\lambda\langle n|m\rangle_\lambda$ holds in the Hilbert space whose basis is spanned by states $|n\rangle_\lambda$.

(c) $a_\lambda^\dagger a|n\rangle_\lambda \equiv \hat{n}_\lambda|n\rangle_\lambda = n|n\rangle_\lambda$; so $\hat{n}_\lambda$ can be thought of as a number operator in the new Fock space. Moreover from this equation we see that $(\hat{n} + \lambda a)|n\rangle_\lambda = n|n\rangle_\lambda$ which indicates simply the ladder operator formalism [Wang(2000)] of the state $|n\rangle_\lambda$.

(d) By equations (11) and (12) we obtain, respectively,

$$a^n|n\rangle_\lambda = \left(\frac{n!}{L_n^{(0)}(-\lambda^2)}\right)^{1/2}|0\rangle_\lambda \tag{14}$$

and

$$|n\rangle_\lambda = \frac{\left(a_\lambda^\dagger\right)^n}{\left(n!L_n^{(0)}(-\lambda^2)\right)^{1/2}}|0\rangle_\lambda. \tag{15}$$

Since $|0\rangle_\lambda = |0\rangle_{\mathrm{SHO}}$, using equations (2) and (3) and the binomial formula we obtain easily the relation between the standard and λ-Fock space as

$$|n\rangle_\lambda = \sum_{m=0}^{n}\frac{(n!)^{1/2}\lambda^{n-m}}{(n-m)!\left(m!L_n^{(0)}(-\lambda^2)\right)^{1/2}}|m\rangle \tag{16}$$

which suggests that every state $|n\rangle_\lambda$ in this non-orthogonal Hilbert space can be regarded as a special finite superposition of $|0\rangle, |1\rangle, \ldots, |n\rangle$ in the standard Fock space $|n\rangle_{\mathrm{SHO}}$ (eliminated indices SHO).

(e) As another point, we see that according to the result of our calculation in equation (16), the states $|n\rangle_\lambda$ are indeed the intermediate number-coherent states denoted by $\|\eta, n\rangle$ which have already been derived by Fu *et al* in [Fu(2000)]

$$|n\rangle_\lambda \equiv \|\eta, n\rangle = D(-\lambda)|\lambda, n\rangle \tag{17}$$

where $\lambda = \sqrt{(1-\eta)/\eta}$ and η is a real probability restricted to $0 < \eta \leqslant 1$, $D(-\lambda)$ is the ordinary displacement operator defined later in equation (34) and

$$|\lambda, n\rangle = \frac{1}{\sqrt{n! L_n^{(0)}(-\lambda^2)}} a^{\dagger^n} |\lambda\rangle \tag{18}$$

is the photon-added coherent state or excited coherent state and $|\lambda\rangle = D(\lambda)|0\rangle$ is a coherent state. In this respect Fu *et al* suggested that the states $\|\eta, n\rangle$ (and also $|n\rangle_\lambda$) are displaced excited coherent states. We are grateful for a scheme suggested in [Fu(2000)] that during an experiment one can generate these states. They have shown that the λ-parameter depends physically on an external driving field (A) in a cavity and the cavity resonant frequency (ω) through the relation $\lambda = A/\omega$.

(f) Using equations (17) and (18) and the BCH lemma, the rhs of equation (16) can be replaced by the compact and interesting formula

$$|n\rangle_\lambda = \frac{\mathrm{e}^{\lambda a}}{\sqrt{L_n^{(0)}(-\lambda^2)}} |n\rangle \equiv T_{n,\lambda}|n\rangle \tag{19}$$

where $T_{n,\lambda}$ is an operator-valued function which exponentially depends on λ and the annihilation operator. The coefficient $\left[L_n^{(0)}(-\lambda^2)\right]^{-1/2}$ is a c-number which keeps the normalizability of the deformed Fock space, and the order of the Laguerre polynomial matches with the old number state which we wish to transform.

(g) Recently Shanta *et al* [Shanta(1994)] and in a systematic manner Man'ko *et al* [Manko(1996)] introduced nonlinear coherent states as eigenfunctions of a deformed annihilation operator $A = af(n)$, such that

$$A|\alpha, f\rangle = \alpha|\alpha, f\rangle \tag{20}$$

in which the representation of these states in $|n\rangle$ basis is given by

$$|\alpha, f\rangle = \mathcal{N}_f(|\alpha|^2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n} f(n)!} |n\rangle \tag{21}$$

where $f(n)$ is the nonlinearity function, $f(n)! \doteq f(0)f(1)\ldots f(n)$ (by convention $f(0) = 1$) and $\mathcal{N}_f$ is an appropriate normalization factor. More recently Re'camier and Jauregui in [Re'camier(2003)] proposed a definition for f-coherent bound states as

$$|\alpha, f\rangle_B = \mathcal{N}_{f,\alpha}^{(m)} \sum_{n=0}^{m} \frac{\alpha^n}{\sqrt{n} f(n)!} |n\rangle \tag{22}$$

where $\mathcal{N}_{f,\alpha}^{(m)}$ is a normalization factor. Following these considerations it is possible to rewrite the states $|n\rangle_\lambda$ in the form of normalized f-coherent bound states

$$|m\rangle_\lambda \equiv \|\eta, m\rangle = \left(\frac{\lambda^{2m} m!}{L_m^{(0)}(-\lambda^2)}\right)^{1/2} \sum_{n=0}^{m} \frac{(\lambda^{-1})^n}{\sqrt{n!}(m-n)!} |n\rangle = |\lambda^{-1}, f\rangle_B. \tag{23}$$

So the space spanned by $|n\rangle_\lambda$ can be regarded as a nonlinear (or f-deformed) Fock space, with the nonlinearity function $f(\hat{n}) = (m - \hat{n})$. It must be noted that the states $|n\rangle_\lambda$ are not eigenstates of the annihilation operator a, because of the finiteness of the upper bound of the summation.

3. Construction of coherent states in $|n\rangle_\lambda$ basis

For constructing the coherent states we begin with the definition of CSs as eigenstates of the annihilation operator [Manko(1996), Das(2002)]

$$a|\alpha,\lambda\rangle = \alpha|\alpha,\lambda\rangle \qquad \alpha \in \mathcal{C} \tag{24}$$

where we call $|\alpha,\lambda\rangle$, λ-CSs. Now if we expand $|\alpha,\lambda\rangle$ in terms of the $|n\rangle_\lambda$ basis

$$|\alpha,\lambda\rangle = \sum_{n=0}^{\infty} C_n |n\rangle_\lambda \tag{25}$$

then by equations (11) and (24) we obtain

$$\sum_{n=0}^{\infty} C_n\sqrt{n}\left(\frac{L_{n-1}^{(0)}(-\lambda^2)}{L_n^{(0)}(-\lambda^2)}\right)^{1/2} |n-1\rangle\lambda = \alpha\sum_{n=0}^{\infty} C_n|n\rangle_\lambda \tag{26}$$

from which we find that the coefficients C_n satisfy the following recurrence relation:

$$C_n = \frac{\alpha^n\left(L_n^{(0)}(-\lambda^2)\right)^{1/2}}{\sqrt{n!}} C_0. \tag{27}$$

The normalization condition of the state $|\alpha,\lambda\rangle$, i.e. $\langle\alpha,\lambda|\alpha,\lambda\rangle = 1$ leads to a complicated series for coefficient C_0 for which after some straightforward but lengthy calculation one obtains

$$C_0 = \exp\left[-\lambda\,\mathrm{Re}(\alpha) - \frac{|\alpha|^2}{2}\right] \tag{28}$$

where $\mathrm{Re}(\alpha)$ is the real part of α. Finally, the normalized λ-CS takes the form

$$|\alpha,\lambda\rangle = \exp\left[-\lambda\,\mathrm{Re}(\alpha) - \frac{|\alpha|^2}{2}\right]\sum_{n=0}^{\infty}\frac{\alpha^n\left(L_n^{(0)}(-\lambda^2)\right)^{1/2}}{\sqrt{n!}}|n\rangle_\lambda \tag{29}$$

which is a '*new representation*' of canonical CS in $|n\rangle_\lambda$ basis. Their inner product which allows overcompleteness is

$$\langle\alpha,\lambda|\beta,\lambda\rangle = \mathcal{N}_{\alpha,\beta,\lambda}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\alpha^{*m}\beta^n\sum_k\frac{\lambda^{2k+m-n}}{k!(n-k)!(m-n+k)!} \tag{30}$$

with

$$\mathcal{N}_{\alpha,\beta,\lambda} = \exp\left[-\lambda\,\mathrm{Re}(\alpha) - \lambda\,\mathrm{Re}(\beta) - \frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2}\right]. \tag{31}$$

Now we imply that by equation (15) the λ-CSs in equation (29) can be expressed in terms of the lowest eigenstate of H_λ as

$$|\alpha,\lambda\rangle = \exp\left[-\lambda\,\mathrm{Re}(\alpha) - \frac{|\alpha|^2}{2}\right]\mathrm{e}^{\alpha a_\lambda^\dagger}|0\rangle. \tag{32}$$

Since $a|0\rangle = 0$, by BCH lemma equation (32) can be rewritten as

$$|\alpha,\lambda\rangle = \mathrm{e}^{-\lambda\,\mathrm{Re}(\alpha)}\exp\left(\alpha a_\lambda^\dagger - \alpha^* a\right)|0\rangle \tag{33}$$

and finally using equation (2) we have

$$|\alpha,\lambda\rangle = \mathrm{e}^{\mathrm{i}\lambda\,\mathrm{Im}(\alpha)}\exp(\alpha a^\dagger - \alpha^* a)|0\rangle = \mathrm{e}^{\mathrm{i}\lambda\,\mathrm{Im}(\alpha)}D(\alpha)|0\rangle \equiv D_\lambda(\alpha)|0\rangle \tag{34}$$

where the imaginary part of α is denoted by $\mathrm{Im}(\alpha)$. Since $D(\alpha)|0\rangle = |\alpha\rangle$ which is the usual CSs, we conclude that $|\alpha,\lambda\rangle$ is identical to $|\alpha\rangle$, up to a phase factor equal to $\mathrm{e}^{\mathrm{i}\lambda\,\mathrm{Im}(\alpha)}$, and therefore $|\alpha, \lambda\rangle = |\alpha\rangle$ whenever $\alpha \in \mathcal{R}$; a result that may be expected from the eigenvalue equation (24). So obviously there is no problem with the resolution of the identity

$$\int \mathrm{d}\mu(\alpha)|\alpha, \lambda\rangle\langle\alpha, \lambda| = I. \tag{35}$$

We would like to emphasize that all we have done in this section is obtaining the explicit form of the canonical CS in a deformed Fock space $|n\rangle_\lambda$, which in this case is non-orthogonal, and we called it the new representation of canonical CS (equation (29)). In fact $|\alpha, \lambda\rangle$ and $|\alpha\rangle$ belong to the same ray in the projective Hilbert space (we will explain more about this result in the appendix). As another result we conclude that by some particular superpositions of displaced excited coherent states $|n\rangle_\lambda$ (which exhibit squeezing [Fu(2000)]), we have obtained canonical CSs.

4. Time evolution of λ-coherent states

We are able to consider the dynamical evolution of the λ-CSs, in the $|n\rangle_\lambda$ basis, which is simply obtained, because the spectrum of H_λ is the same as H_{SHO}

$$\begin{aligned} U(t)|\alpha, \lambda\rangle &= \exp\left[-\lambda\,\mathrm{Re}(\alpha) - \frac{|\alpha^2|}{2}\right]\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left(L_n^{(0)}(-\lambda^2)\right)^{1/2}\mathrm{e}^{-\mathrm{i}H_\lambda t}|n\rangle_\lambda \\ &= \exp\left[-\lambda\,\mathrm{Re}(\alpha) - \frac{|\alpha^2|}{2}\right]\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left(L_n^{(0)}(-\lambda^2)\right)^{1/2}\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})t}|n\rangle_\lambda \\ &= \mathrm{e}^{-\mathrm{i}t/2}|\alpha(t), \lambda\rangle \end{aligned} \tag{36}$$

where we have used $\alpha(t) \equiv \alpha\,\mathrm{e}^{-\mathrm{i}t}$. This means that the time evolution of λ-CSs in this non-orthogonal basis remains coherent for all times (*temporal stability*).

5. Statistical properties of $|\alpha, \lambda\rangle$ in $|n\rangle_\lambda$ bases

For standard CS $|\alpha\rangle$, the occupation number distribution is Poissonian

$$P(n) = |\langle n|\alpha\rangle|^2 = \mathrm{e}^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} \tag{37}$$

whose mean and variance are equal to $|\alpha|^2$. Similarly in the case of our λ-CSs, we define

$$P_\lambda(m) = |{}_\lambda\langle m|\alpha, \lambda\rangle|^2 = \frac{m!}{L_m^{(0)}(-\lambda^2)}\mathrm{e}^{-|\alpha|^2}\sum_{n=0}^{m}\sum_{p=0}^{m}\frac{\lambda^{2m-n-p}\alpha^{*p}\alpha^n}{p!(m-p)!n!(m-n)!} \tag{38}$$

as the probability of finding the states $|\alpha, \lambda\rangle$ in the $|n\rangle_\lambda$ basis. It is easy to verify that $\lim_{\lambda\to 0} P_\lambda(m) = P(m)$. While the distribution in the $|n\rangle_{\mathrm{SHO}}$ basis is Poissonian, this is not so in the $|n\rangle_\lambda$ basis. As usual we have

$$_\lambda\langle\hat{m}\rangle_\lambda = \sum_m m P_\lambda(m) \qquad {}_\lambda\langle\hat{m}^2\rangle_\lambda = \sum_m m^2 P_\lambda(m). \tag{39}$$

To check Poissonian, sub-Poissonian or super-Poissonian statistics, we can evaluate the Mandel parameter defined as

$$Q(\alpha, \lambda) = \frac{\langle\hat{m}^2\rangle - \langle\hat{m}\rangle^2}{\langle\hat{m}\rangle} - 1. \tag{40}$$

We see from figure 1 that the states $|\alpha, \lambda\rangle$ exhibit sub-Poissonian (nonclassical) or super-Poissonian statistics in the $|n\rangle_\lambda$ basis depending on the values of α. While it is seen that for

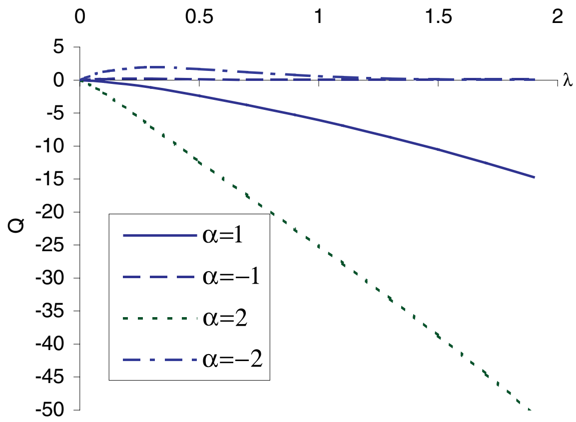

Figure 1. Mandel parameter for λ-CSs in the $|n\rangle_\lambda$ basis as a function of λ for different values of α. α is taken to be real.

$\alpha = 1, 2$ the state is sub-Poissonian, when $\alpha = -1, -2$ it is super-Poissonian. As another feature, our numerical results show that when $\alpha \in \mathcal{R}$ and $\alpha < 0$ for large values of λ, the statistical behaviour of the state is nearly close to Poissonian.

6. Squeezed states in terms of $|n\rangle_\lambda$-bases

According to the statement of Solomon and Katriel [Solomon(1990)], the conventional squeezed states are obtained by the action of a linear combination of creation and annihilation operators on an arbitrary state. Now by generalizing this procedure to $a_\lambda^\dagger = a^\dagger + \lambda I$ and $a_\lambda = a$ of the deformed oscillator algebra with $a, a_\lambda^\dagger$ as annihilation and creation operators we have

$$\left(a - \xi a_\lambda^\dagger\right)|\xi, \lambda\rangle = 0 \qquad \xi \in \mathcal{C}. \tag{41}$$

This equation for $\lambda = 0$ leads to the squeezed vacuum states

$$|\xi\rangle = C_0 \sum_{n=0}^{\infty} \xi^n \left[\frac{(2n-1)!!}{(2n)!!}\right]^{\frac{1}{2}} |2n\rangle_\lambda = C_0 S(\xi)|0\rangle \tag{42}$$

where the normalization coefficient may be determined as

$$C_0 = \left[\sum_{n=0}^{\infty} \frac{|\xi|^{2n}(2n-1)!!}{(2n)!!}\right]^{-\frac{1}{2}} \tag{43}$$

and $S(\xi) = \exp\left[\frac{\xi (a^\dagger)^2}{2}\right]$ is the squeezed operator. But in general from equation (41) and by the same procedure followed above and in section 2, for $\lambda \neq 0$ we will arrive at a '*new representation*' for squeezed states (λ-SSs) as

$$|\xi, \lambda\rangle = C_0 \sum_{n=0}^{\infty} \xi^n \left(L_{2n}^{(0)}(-\lambda^2)\right)^{\frac{1}{2}} \sqrt{\frac{(2n-1)!!}{(2n)!!}} |2n\rangle_\lambda \qquad \xi \in \mathbf{D} \tag{44}$$

where $\mathbf{D} = \{\xi \in \mathcal{C} | \quad |\xi| < R\}$ is a disc centred at the origin in the complex ξ plane with radius R, which by a suitable transformation can be transformed to a unit disc. For the normalization factor one may obtain

$$C_0 = \left[\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \xi^n \xi^{*m} (2n-1)!!(2m-1)!! \sum_k \frac{\lambda^{2k+2m-2n}}{k!(2n-k)!(2m-2n+k)!} \right]^{-\frac{1}{2}}. \tag{45}$$

These λ-SSs are normalizable provided that the coefficient C_0 is nonzero and finite. It is very hard, if not impossible, to obtain the analytic form of the radius of convergence for C_0. This is due to the presence of three sigmas and two distinct parameters λ and ξ in the series of equation (45). But this has been checked on the disc $\mathbf{D}$ for a large value of λ and we obtained a finite radius of convergence for C_0. But our investigations on numerical results show that as λ is increased, the radius of convergence is reduced and this will confine the freedom of choosing λ.

We see that these states are also even but in the $|n\rangle_\lambda$ basis, although if we transform them by equation (16) to the ordinary basis $|n\rangle$, no definite parity can be observed. To change this representation of squeezed states, equation (44), to the old standard Fock states, we obtain

$$|\xi, \lambda\rangle = C_0 \exp\left[\frac{\xi \left(a_\lambda^\dagger\right)^2}{2}\right] |0\rangle = C_0 \exp\left[\frac{\xi \lambda^2}{2}\right] S(\xi) D(\xi\lambda) |0\rangle \tag{46}$$

where as defined before $S(\xi)$ and $D(\xi\lambda)$ are squeezed and displacement operators, respectively. The state $|\xi, \lambda\rangle$ in the standard Fock space is the one which is named as the squeezed coherent state. From the above discussion we immediately conclude that translating $|\xi, \lambda\rangle$ in the $|n\rangle_\lambda$ basis to a state in the standard Fock space $|n\rangle$ does not coincide with that referred to in equation (42). This is due to the fact that in obtaining the SSs, both the annihilation and deformed creation operators contribute (equation (41)). To check the nonclassical properties of these states first we investigate the quadrature squeezing. The quadratures x and p are related to a and $a^\dagger$ according to

$$\hat{x} = \frac{a + a^\dagger}{\sqrt{2}} \qquad \hat{p} = \frac{a - a^\dagger}{\mathrm{i}\sqrt{2}}. \tag{47}$$

As usual the variances of coordinates x and momentum p are as follows:

$$(\triangle x)^2 = \langle x^2\rangle - \langle x\rangle^2 = \tfrac{1}{2}[1 + \langle a^2\rangle + \langle a^{\dagger 2}\rangle + 2\langle a^\dagger a\rangle - \langle a\rangle^2 - \langle a^\dagger\rangle^2 - 2\langle a\rangle\langle a^\dagger\rangle] \tag{48}$$

$$(\triangle p)^2 = \langle p^2\rangle - \langle p\rangle^2 = \tfrac{1}{2}[1 - \langle a^2\rangle - \langle a^{\dagger 2}\rangle + 2\langle a^\dagger a\rangle + \langle a\rangle^2 + \langle a^\dagger\rangle^2 - 2\langle a\rangle\langle a^\dagger\rangle] \tag{49}$$

where all of the expectation values should be calculated with respect to the states $|\xi, \lambda\rangle$. All we need to evaluate the rhs of equations (48) and (49) are the terms such as

$${}_\lambda\langle m|\left(a_\lambda^\dagger\right)^k|n\rangle_\lambda = (n+k)! \left(\frac{m!}{n! L_n^{(0)}(-\lambda^2) L_m^{(0)}(-\lambda^2)}\right)^{\frac{1}{2}} \sum_l \frac{\lambda^{(2l+m-n-k)}}{l!(n+k-l)!(m-n-k+l)!} \tag{50}$$

$${}_\lambda\langle m|a^k|n\rangle_\lambda = \left(\frac{m!n!}{L_n^{(0)}(-\lambda^2) L_m^{(0)}(-\lambda^2)}\right)^{\frac{1}{2}} \sum_l \frac{\lambda^{(2l+m-n+k)}}{l!(n-k-l)!(m-n-k+l)!} \tag{51}$$

and

$$\begin{aligned} {}_\lambda\langle m|\left(a_\lambda^\dagger\right)^r a^k|n\rangle_\lambda = {} & \frac{(n-k+r)!}{(n-k)!} \left(\frac{m!}{L_m^{(0)}(-\lambda^2) L_n^{(0)}(-\lambda^2)}\right)^{\frac{1}{2}} \\ & \times \sum_l \frac{\lambda^{2l+m-n+k-r}}{l!(n-k+r-l)!(m-n+k-r+l)!}. \end{aligned} \tag{52}$$

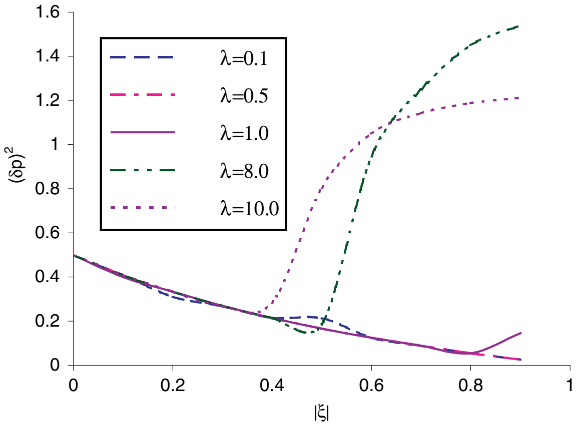

Figure 2. Uncertainty in field quadrature p, $(\delta p)^2$, as a function of $|\xi|$ for different values of λ.

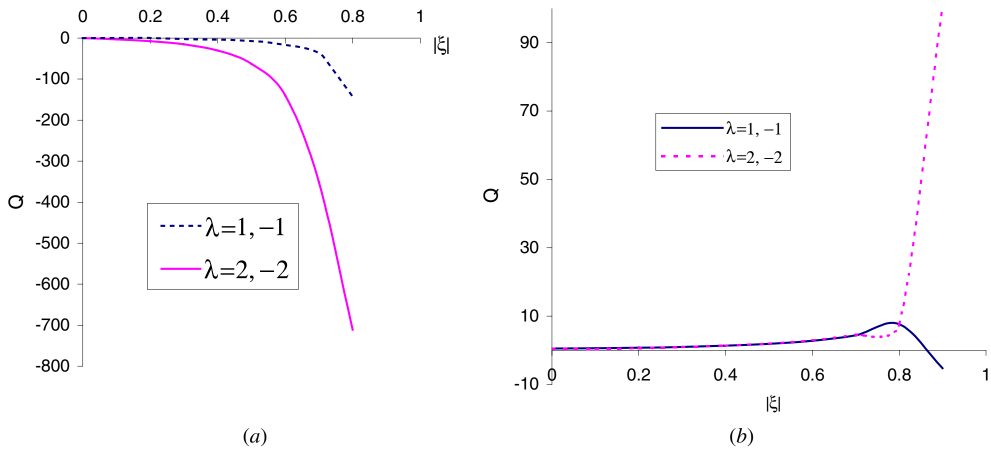

Figure 3. (*a*) Mandel parameter of λ-SSs as a function of $|\xi|$ for different values of λ in the $|n\rangle_\lambda$ basis. (*b*) Mandel parameter of λ-SSs as a function of $|\xi|$ for different values of λ in the $|n\rangle$ basis.

Figure 2 shows that squeezing occurs in both p- and x-quadratures depending on the values of ξ and λ. For $\lambda \leqslant 1$ only the p-quadrature is squeezed over all ranges of $|\xi|$ (except near $\xi = 1$) and the strength of squeezing is not very sensible to the value of λ but depends on the values of $|\xi|$. By increasing the λ-parameter, the squeezing transfers from the p- to the x-quadrature very fast with respect to the horizontal axis ($|\xi|$). Figures 3(*a*) and (*b*) show the Mandel parameter, equation (40), as a function of $|\xi|$ for different values of λ in two bases $|n\rangle$ and $|n\rangle_\lambda$. According to these results, while the λ-SSs, equation (44), have sub-Poissonian (nonclassical) behaviour in the $|n\rangle_\lambda$ basis (figure 3(*a*)), they have super-Poissonian statistics in the $|n\rangle_{\mathrm{SHO}}$ basis (figure 3(*b*)).

7. Conclusion

The states proposed by Beckers *et al* [Beckers(1998)] (or by Fu *et al* [Fu(2000)]) $|n\rangle_\lambda$ are not coherent ones, but they exhibit squeezing. We showed that these states can be regarded as f-deformed coherent bound states and demonstrated that our infinite-dimensional Hilbert space may be spanned by them. Our motivation for this consideration is the greater generality and flexibility of the non-orthogonal basis $\{|n\rangle_\lambda, n \in \mathbb{N}\}_{n=0}^{\infty}$ than the orthogonal one $\{|n\rangle, n \in \mathbb{N}\}_{n=0}^{\infty}$. Then we concluded that by some special superposition of the deformed Fock space ($|n\rangle_\lambda$), we can obtain the representations of coherent states $|\alpha, \lambda\rangle$ as well as squeezed coherent states $|\xi, \lambda\rangle$ in the new bases. In comparison with our new nonlinear–nonorthogonal but normalized Fock space with basis $|n\rangle_\lambda$ and the orthonormal basis $|n\rangle$ of the old Fock space, it is worth noting that the one we consider is not just a change of basis. By this we mean that there is a cut-off in the sum on the rhs of relation (16). In other words the sum does not go on to infinity. So it is not surprising that in the new Fock space we gain a set of new physical aspects. It will be interesting to study other general systems and see if their Fock spaces can be constructed by other superpositions of number states and then derive new coherent and squeezed states.

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Appendix. How different deformations affect coherent states

Here we bring some examples to show that how the deformation of creation, annihilation or both for a specific dynamical system affects the produced CSs. We construct the Hamiltonian

$$\begin{aligned} H' &= H_{\mathrm{SHO}}^2 = \left(a^\dagger a + \tfrac{1}{2}\right)^2 \\ &= a^\dagger(a^\dagger a + 2)a + \tfrac{1}{4} \\ &= a^\dagger(\hat{n} + 2)a + \tfrac{1}{4}. \end{aligned} \tag{A1}$$

Before discussing the coherent states of this nonlinear Hamiltonian, let us give some physical aspects for it. First it is obvious that $H'|n\rangle = \left(n + \frac{1}{2}\right)^2|n\rangle$ which means that the states are not equally spaced. We know that a number of quantum mechanical systems, e.g. square-well [Styer(2001)], Morse potential [Morse(1926)] and Pöschl–Teller potential [Posh(1933)], exhibit quadratic spectra $E_n \sim n^2$, which indicate the nonlinear nature of them. Secondly we bring briefly the physical meaning of the above Hamiltonian as Man'ko *et al* proposed [Manko(1997)]. For this purpose we may apply the inverse procedure they proposed, i.e. replacing creation and annihilation operators by complex variables α and α^*, after all we have

$$H'(\alpha, \alpha^*) = (2 + \alpha\alpha^*)\alpha\alpha^* \tag{A2}$$

where $\alpha = (x + \mathrm{i}y)/\sqrt{2}$ and $\alpha^* = (x - \mathrm{i}y)/\sqrt{2}$, $\{x, y\} = 1$ and by $\{\cdot, \cdot\}$ we mean the Poissonian bracket. Therefore we obtain

$$\dot{\alpha} + \mathrm{i}(2 + 4\alpha\alpha^*)\alpha = 0. \tag{A3}$$

Since $\alpha\alpha^*$ is a constant of motion we conclude that the frequency of this nonlinear Hamiltonian dynamics is

$$\omega = 2(1 + 2\alpha\alpha^*). \tag{A4}$$

This shows the dependence of the new frequency on the number of particles, which is a nonlinear phenomena called by Man'ko and Tino '*the blue shift of the frequency of light*' [Manko(1995)].

Now we try to construct CSs with three different deformations imposed on this Hamiltonian as follows:

(I) $A = f_1(\hat{n})a$, $A^\dagger = a^\dagger$ where $f_1(\hat{n}) = \hat{n} + 2$. By the known procedure we have

$$|\alpha, f_1\rangle = C_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{(n!)^{\frac{3}{2}}}|n\rangle \qquad C_0 = \left[\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{(n!)^3}\right]^{-\frac{1}{2}}. \tag{A5}$$

(II) $B = af_2(\hat{n})$, $B^\dagger = a^\dagger f_2(\hat{n})a$ where $f_2(\hat{n}) = \sqrt{\hat{n}+2}$. In this case we obtain

$$|\alpha, f_2\rangle = C_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}|n\rangle \qquad C_0 = \left[\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{(n!)^2}\right]^{-\frac{1}{2}}. \tag{A6}$$

(III) $C = a$, $C^\dagger = a^\dagger f_3(\hat{n})$, $f_3(\hat{n}) = \hat{n} + 2$ where it is really expected that by C operator we demand the usual canonical coherent states. Therefore, in conclusion it is not surprising that when we translate the λ-CSs, equation (29), in the standard Fock space we obtain the canonical CS, clearly because we did not change the form of the annihilation operator. As a result we observe that different deformations lead to different CSs. The first two of these CSs (and obviously the third) are special kinds of generalized CSs proposed by Penson and Solomon [Penson(2001)] in the form

$$|Z\rangle_C = \mathcal{N}_C(|Z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{Z^n}{\sqrt{C(n)}}|n\rangle \qquad Z \in \mathcal{C} \tag{A7}$$

where the normalization factor is

$$\mathcal{N}_C(|Z|^2) = \sum_{n=0}^{\infty} \frac{|Z|^{2n}}{C(n)}. \tag{A8}$$

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